Some Basic Aspects of Elastic Wave Equation in General Media

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1. Introduction
2. Wave propagation in Continuum media
3. Basic notions on Elasticity theory (Marsden.J)

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2. Wave propagation in Continuum media
3. Basic notions on Elasticity theory (Marsden.J)
4. An example and further works

Wave Propagation in Continuum Media

- Hook‘s law
- Cauchy's equations of motion
- Wave equation for P-waves in homogeneous and isotropic media
- Wave equation for S-waves in homogeneous and isotropic media

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Wave Propagation in Continuum Media

- Hook‘s law

$$\sigma_{ij} = \sum_{k,l} \mathcal{C}_{ijkl} \; \epsilon_{kl}$$

where

$$\begin{aligned} \sigma_{ij} &: \quad \text{is the strain tensor,} \\ \mathcal{C}_{ijkl} &: \quad \text{is the stiffnes tensor,} \\ \epsilon_{kl} &: \quad \text{is the stress tensor.} \end{aligned}$$

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  From the balance of momentum one gets

$$\rho(\vec{x})\frac{\partial^2 \vec{u}_i}{\partial t^2} = \sum_j \frac{\partial}{\partial x_j}\sigma_{ij}$$

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  For an Isotropic media

$$\sigma_{ij} = \lambda\delta_{ij}\sum_k \epsilon_{kk} + 2\mu\epsilon_{ij}$$

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then

$$\rho(\vec{x})\frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu)[\triangledown(\triangledown \cdot \vec{u})] + \mu \triangledown^2 \vec{u}$$

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In general curvilinear coordinates

$$\bigtriangledown^2 \vec{u} = \bigtriangledown(\bigtriangledown \cdot \vec{u}) - \bigtriangledown \times (\bigtriangledown \times \vec{u})$$

and defining

$$\begin{aligned} \varphi &= \bigtriangledown \cdot \vec{u} \\ \psi &= \bigtriangledown \times \vec{u} \end{aligned}$$

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  we get

$$\rho(\vec{x})\frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + 2\mu) \bigtriangledown \varphi - \mu \bigtriangledown \times \psi$$

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$$\nabla^2\varphi - \frac{1}{v_p^2}\frac{\partial^2\varphi}{\partial t^2} = 0$$

where

$$v_p = \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}}$$

- Wave equation for S-waves in homogeneous and isotropic media

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$$\bigtriangledown^2\psi - \frac{1}{v_s^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

where

$$v_s = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$$

On Wave equation

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Consider the IVP

$$\begin{aligned}
\nabla^2\vec{u} - \frac{1}{v^2}\frac{\partial^2\vec{u}}{\partial t^2} &= 0 \\
\vec{u}(\vec{x},0) &= \gamma(\vec{x}) \\
\frac{\partial\vec{u}}{\partial t}|_{t=0} &= \eta(\vec{x})
\end{aligned}$$

On Wave equation

- In one dimension (1-D)

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$$u(x,t) = \frac{1}{2}\left[\gamma(x+vt) + \gamma(x-vt) + \frac{1}{v}\int_{x-vt}^{x+vt} \eta(s)ds\right]$$

where

$$\begin{aligned}\gamma(x) &= f(x) + g(x) \\ \eta(x) &= v[f'(x) + g'(x)]\end{aligned}$$

for some $f, g \in \mathcal{C}^2(\Omega)$

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- In two dimensions (2-D)

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$$\begin{aligned}\vec{u}(\vec{x},t) &= \frac{d}{dt}\left[\frac{4\pi^2}{v}\iint_{D(\vec{x},vt)}\frac{\gamma(s_1,s_2)}{\sqrt{(vt)^2-[(s_1-x_1)^2+(s_2-x_2)^2]}}ds_1\,ds_2\right]\\ &+ \frac{4\pi^2}{v}\iint_{D(\vec{x},vt)}\frac{\eta(s_1,s_2)}{\sqrt{(vt)^2-[(s_1-x_1)^2+(s_2-x_2)^2]}}ds_1\,ds_2\end{aligned}$$

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- *The earth is at least a visco elastic medium, in which absorption losses give rise to attenuation and dispersion effects.*
- *The elastic wave equation is framed in terms of tensor operators acting on vector quantities.*

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- *The earth is at least a visco elastic medium, in which absorption losses give rise to attenuation and dispersion effects.*
- *The elastic wave equation is framed in terms of tensor operators acting on vector quantities.*
- *...it is also true that a proper treatment of anisotropy fundamentally demands an elastic viewpoint, even when only P-waves (quasi-P waves) are contemplated.*

Elasticity Theory

Elasticity Theory

- A configuration on $\mathcal{B}$ is a smooth, orientation preserving and invertible mapping

$$\Phi : \mathcal{B} \to \int$$

  The set of all configurations of $\mathcal{B}$ is denoted $\mathcal{C}$
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- The material velocity and acelerations are defined as (for X fixed)

$$\begin{aligned} V_t(X) &= \frac{\partial}{\partial t}\Phi(X, t) \\ A_t(X) &= \frac{\partial}{\partial t}V_t(X) \end{aligned}$$

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- The spatial velocity and acelerations are defined as (for t fixed)

$$\begin{aligned} v_t &:= V_t \circ \mathit{\Phi}^{-1} \\ a_t &:= A_t \circ \mathit{\Phi}^{-1} \end{aligned}$$

Elasticity Theory

Elasticity Theory

- The deformation gradient, is given by

$$\begin{aligned} F : T\mathcal{B} &\to T\mathcal{S} \\ F(X, W) &= (\Phi(X), D\Phi(x) \cdot W) \end{aligned}$$

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Elasticity Theory

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- The right Cauchy-Green tensor is given by

$$
\begin{aligned}
C : T_X\mathcal{B} &\to T_X\mathcal{B} \\
C(X, W) &= \left(X, D\Phi(X)^T D\Phi(X) \cdot W\right) \\
C(X) &= F^T(X)F(X)
\end{aligned}
$$

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- some properties of C
  1. C is Symmetric
  2. C is semi-positive definite
  3. If every F is one-to one, then C is positive definite and invertible.
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- The left Cauchy-Green tensor is given by

$$\begin{aligned} b : T_x\Phi(\mathcal{B}) &\to T_x\Phi(\mathcal{B}) \\ b(x) &= F(X)F^T(X) \end{aligned}$$

-

Elasticity Theory

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Elasticity Theory

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Elasticity Theory

- Consider the symmetric, positive definite, linear transformations U, V such that

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\begin{aligned}
U^2 &= C \\
V^2 &= b
\end{aligned}
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Elasticity Theory

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- It can be shown that (polar decomposition of F)

$$F = RU = VR$$

for some unique orthogonal transform

$$R : T_X\mathcal{B} \to T_x\mathcal{S}$$

and

$$U = R^T VR$$

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and

$$U = R^T VR$$

- The Strain tensor is given by

$$\begin{aligned} E : T\mathcal{B} &\to T\mathcal{B} \\ E &= \frac{1}{2}[C - Id] \end{aligned}$$

An example. Yasutomi. Y

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An example. Yasutomi. Y

- For small motions of $\mathcal{B}$, we have

$$\Phi_t^i(X) = x^i + u^i(X, t)$$

where $u = \sum u^i(X, t)\partial_i$ is the displacement vector field.
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An example. Yasutomi. Y

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- The strain tensor ε_{ij} is given by

$$\varepsilon_{ij} dx^i \otimes dx^j = \frac{1}{2}\left[*ds(X)^2 - ds(X)^2\right]$$

  , then

$$\varepsilon_{kl} = \frac{1}{2}\left(g_{km}\partial_l u^m + g_{ml}\partial_k u^m + u^m \partial_m g_{kl}\right)$$

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An example. Yasutomi. Y

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- Since

$$\begin{aligned} \frac{\sigma_{ij}}{\sqrt{|g|}} &= C_{ijkl}\varepsilon_{kl} \\ C_{ijkl} &= \lambda g^{ij} g^{kl} + \mu g^{ik} g^{jl} + \mu g^{il} g^{jk} \\ df^{i} &= \frac{\sigma_{ij}}{\sqrt{|g|}} dS_j \end{aligned}$$

we have, for an elastic, homogeneous and isotropic body, the equation:

$$\rho \partial_{tt} u^i = \lambda g^{ij} \bigtriangledown_j \bigtriangledown_k u^k + \mu g^{jk} \bigtriangledown_j \bigtriangledown_k u^i + \mu g^{ik} \bigtriangledown_j \bigtriangledown_j u^j$$

Further Works

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- To stablish the motion equations, derived from conservation principles, for different configurations which induce the symmetry of the medium.
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- To decompose the above equations via diagonal operators defined on the body manifold.
- Wave field extrapolation

Some references

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